Characteristics of the wave function of coupled oscillators in semiquantum chaos

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Using the method of adiabatic invariants and the Born-Oppenheimer approximation, we have calculated the ground-state wave function of a pair of coupled oscillators in so-called semiquantum chaos. Some interesting characteristics, e.g., the similarities and differences between the wave functions in the regular and chaotic states have been found. Time-correlation functions of the wave functions and their Fourier spectra in two states have also been investigated. The sensitivity of the wave function in the chaotic state to the initial conditions has been identified.

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I. INTRODUCTION

As a topic of research, chaos has long been investigated in classical systems. In comparison with classical chaos in dynamic systems, chaos in a quantum system has not been properly defined despite much research work. In fact, from the obvious correspondence between classical and quantum mechanics, the classical chaos in some dynamical systems must have its counterpart in the corresponding quantum systems. Percival [1] found that under semiclassical approximation, the quantum energy spectrum of an N-dimensional conservative system can be divided into two parts, regular and irregular, which relate, respectively, to periodic and chaotic motions in the classical phase space. Bohigas [2] and Berry [3] then developed the approach of energy-level statistics to describe the properties of regular and irregular quantum energy spectra. Meanwhile, the studies of morphology of eigenfunctions made by Shapiro [4] and Heller [5] give us information about regular and irregular states of quantum systems. Recently, there have been some counterpart systems such as quantum billiards, quantum spin systems and so on, found in the field of condensed-matter physics.

We all know that classical chaos is characterized by the sensitivity of the system to the initial conditions (or in mathematics, by the positive Lyapunov characteristic exponent). But in quantum mechanics, nonexistence of the concept of path or trajectory prevents us from studying those irregular behaviors considered as quantum chaos by using similar methods in studying classical chaos. Quite a few of these correspondences have been thoroughly studied and many approaches, e.g., the random matrix theory, the orbit scar theory and so on, have been discovered to study properties of these kind of irregular behaviors, or the so-called ''quantum chaos.'' In these approaches, many researchers have focused on the concept of integrability in quantum mechanics and hope to get an exact definition of quantum chaos. Others want to find the manifestation of quantum chaos in actual experimental systems.

Cooper *et al.* [6] considered a system in which a classical oscillator interacts with a purely quantum-mechanical oscillator. The complete dynamics of the coupled quantum and classical oscillators was described by a *classical effective Hamiltonian*, which is the expectation value of the quantum Hamiltonian. They found that the effective classical Hamiltonian totally determines the time evolution of the quantum oscillator, and shows chaotic behavior due to coupling between the two oscillators, leading the parameters describing the quantum-mechanical wave function and expectation values to be sensitive to initial conditions. This phenomenon was named *semiquantum chaos* by its discoverers, Cooper *et al*. [6]. Faccioli *et al*. in their comment [7] gave a more proper treatment for this problem.

Though the proper semiclassical evolution functions of the system have been obtained in their work, both Cooper *et al*. and Faccioli *et al*. have only paid their attention to the *average value* of the time-dependent occupation number in the chaotic state, and the *wave function* of the quantum oscillator in the coupled system has not been cared for. As is well known, in quantum mechanics the wave function, especially the ground-state wave function, can show not only the static but also the dynamical information of the system. In consideration of the above insufficiency, we here will treat the same coupled system as one in Cooper's paper [6] that also consists of a pair of coupled oscillators, one considered as a classical oscillator and the other a quantum one. However, differing from both of Cooper *et al*. [6] and Faccioli *et al*., [7] we will try to find the ground-state *wave functions* of the system in both chaotic and regular states and then discuss some interesting properties of them in this paper. In Sec. II, we will describe methods used to calculate the wave function, i.e., the method of adiabatic invariants [8] and the Born-Oppenheimer approximation [9–12]. In Sec. III, the wave function of the quantum system concerned here is obtained; and in Sec. IV, we will discuss evolutional characteristics of the system. Because the evolution functions of the system are obtained by the semiclasical treatments used by Cooper *et al*. and Faccioli *et al*. in their papers, it is not difficult to choose a chaotic state together with a regular one in order to compare the differences between the ground-state wave functions of the system in these two states. Finally, we will end the article with some conclusions in Sec. V.

II. METHOD OF ADIABATIC INVARIANTS

A. General feature

For a system whose Hamiltonian is an explicit function of time, we can assume existence of an explicitly time-dependent Hermitian invariant operator $I(t)$, defined as

$$\frac{dI}{dt} \equiv \frac{\partial I}{\partial t} + \frac{1}{i\hbar}[I,H] = 0. \tag{1}$$

We denote the eigenvalues of I by λ, and its orthonormal eigenstate by $|\lambda,\mu\rangle$, where μ represents all of the quantum numbers except λ.

Then, a new set of eigenvectors of $I(t)$ related to the initial set $|\lambda,\mu\rangle$ can be defined by a time-dependent gauge transformation

$$|\lambda,\mu\rangle_\alpha = e^{i\alpha_{\lambda\mu}(t)}|\lambda,\mu\rangle, \tag{2}$$

where the $\alpha_{\lambda\mu}(t)$ is an arbitrary real function of time. The method of adiabatic invariant tells us if the phases $\alpha_{\lambda\mu}(t)$ satisfy the following condition:

$$\hbar\frac{d\alpha_{\lambda\mu}}{dt} = \langle\lambda,\mu|i\hbar\frac{\partial}{\partial t} - H|\lambda,\mu\rangle, \tag{3}$$

we can obtain a new set of eigenstates of $I(t),|\lambda,\mu\rangle_\alpha$, obeying the Schrödinger equation.

B. A time-dependent harmonic oscillator

For certain type of time-dependent harmonic oscillators, their hamiltonian can be written as

$$H(t) = \frac{1}{2M}[p^2 + \Omega^2(t)q^2], \tag{4}$$

where q is a canonical coordinate, p is its conjugate momentum, $\Omega(t)$ is an arbitary, piecewise-continuous function of time, and M is a real, positive mass parameter.

It can be easily found that for Eq. (4), its corresponding invariant operator $I(t)$ is equal to

$$I(t) = \frac{1}{2}\left[\frac{1}{\rho^2}q^2 + (M\dot{\rho}q - \rho p)^2\right], \tag{5}$$

and the new variant ρ satisfies

$$M^2\ddot{\rho} + \Omega^2\rho = \frac{1}{\rho^3}. \tag{6}$$

Time-dependent canonical lowering and raising operators a and $a^\dagger$ can be introduced as follows:

$$a = (2\hbar)^{-1/2}\left[\frac{q}{\rho} + i(\rho p - M\dot{\rho}q)\right], \tag{7}$$

$$a^\dagger = (2\hbar)^{-1/2}\left[\frac{q}{\rho} - i(\rho p - M\dot{\rho}q)\right], \tag{8}$$

which obey the canonical commutation rule $[a,a^\dagger]=1$. Then, the invariant $I(t)$ can be given in terms of a and $a^\dagger$ as $I=\hbar(a^\dagger a+\frac{1}{2})$. If the eigenstates and eigenvalues of the operator $a^\dagger a$ are denoted by $|s\rangle$ and s, i.e., $a^\dagger a|s\rangle = s|s\rangle$, $s=0,1,2,\ldots$, we will have the following relations $a|s\rangle = s^{1/2}|s-1\rangle, a^\dagger|s\rangle = (s+1)^{1/2}|s+1\rangle$. Thus, the eigenvalue spectrum of I is given by $\lambda_s = (s+\frac{1}{2})\hbar$, $s=0,1,2,\ldots$.

If we want to get the eigenvalue and eigenstate of the original time-dependent Hamiltonian by means of the invariant, we must calculate the factor $\alpha_{\lambda\mu}(t)$ in the gauge transformation. For the eigenstate $|s\rangle$, it can be easily proved that

$$\alpha_{\lambda\mu}(t) = \alpha_s(t) = -\frac{1}{M}\left(s+\frac{1}{2}\right)\int^t dt'\frac{1}{\rho^2(t')}. \tag{9}$$

III. SEMICLASSICAL DESCRIPTION OF THE COUPLED HARMONIC OSCILLATORS SYSTEM

In this paper, we mainly deal with a coupled system, composed of a classical oscillator and a purely quantum oscillator, which is described by a Hamiltonian [6]

$$H = \frac{1}{2}p^2 + \frac{1}{2}\Pi_A^2 + \frac{1}{2}(m^2 + e^2A^2)x^2, \tag{10}$$

where $p(t)=\dot{x}(t)$ and $\Pi_A = \dot{A}(t)$. The coordinates x and A describe, respectively, the motion of the quantum oscillator and the classical one in the system. Using the Born-Oppenheimer approximation, we can decouple the system into a classical part and a quantum part, each of which can be handled easily.

From Hamiltonian (10), a Schrödinger equation for the system with energy E is obtained

$$\frac{1}{2}\left(-\hbar^2\frac{\partial^2}{\partial x^2} + \omega^2x^2 - \hbar^2\frac{\partial^2}{\partial A^2} - 2E\right)\Psi_E(x,A) = 0, \tag{11}$$

where $\omega^2 = m^2 + e^2A^2$. After factorizing $\Psi_E(x,A) = \psi(A)\chi(x,A)$, and following the treatment in Ref. [7], i.e., using the semiclassical approximation, we obtain the following coupled equations for the $\psi(A)$ and $\chi(x,A)$:

$$\left(\frac{1}{2}\dot{A}^2 + \langle\hat{H}_x\rangle\right)\psi = E\psi \tag{12}$$

and

$$\left(\hat{H}_x - i\hbar\frac{\partial}{\partial t}\right)\chi_s = 0, \tag{13}$$

where

$$\hat{H}_x = -\frac{1}{2}\left(\hbar^2\frac{\partial^2}{\partial x^2} + \omega^2x^2\right), \tag{14}$$

$$\chi(x,A) = \chi_s(x,A)\exp\left[\int^t dt'\left(\frac{i}{\hbar}\langle\hat{H}_x\rangle + \left\langle\frac{\partial}{\partial A'}\right\rangle\dot{A}'\right)\right], \tag{15}$$

and the average value of an operator $\langle\hat{O}\rangle$ is defined as

$$\langle\hat{O}\rangle = \frac{\int dx\chi^*\hat{O}\chi}{\int dx\chi^*\chi}. \tag{16}$$

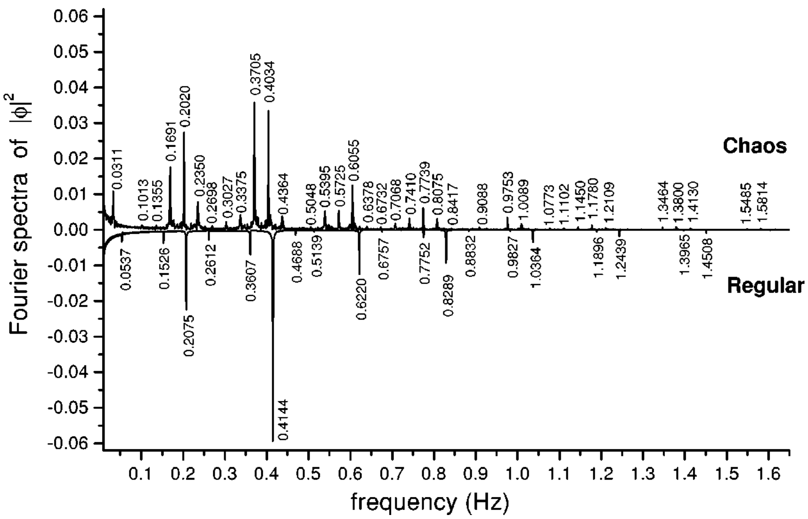

FIG. 1. Fourier spectra of the time evolution of the probability density, $|\phi(x,G(t))|^2$, in chaotic and regular states.

Now using the adiabatic invariant method discussed in Sec. II, it is easy to solve the Schrödinger equation, Eq. (13), for the quantum part of the system by introducing an adiabatic invariant $I(t)$, and the time-dependent canonical lowering and raising operators a and $a^\dagger$ in Eq. (5), (7) and (8). In order to compare with the paper of Cooper *et al.* paper, we substitute ρ in those equations for G, i.e., let $\rho=(2G)^{1/2}=(2\langle x^2\rangle)^{1/2}$. On defining the ground state of a by $a|0\rangle=0$, we can calculate the average values of x^2, $\dot{x}^2$, and the Hamiltonian H_x. At the same time, the equations of A and ρ are changed into the following form:

$$\ddot{A}=-\frac{\partial}{\partial A}\langle\hat{H}_x\rangle=-\hbar e^2 AG, \tag{17}$$

$$\frac{1}{2}\frac{\ddot{G}}{G}-\frac{1}{4}\left(\frac{\dot{G}}{G}\right)^2-\frac{1}{4G^2}+1+e^2A^2=0. \tag{18}$$

In the coordinate representation, equation $a|0\rangle=0$ gets the form of

$$\left(\frac{\partial}{\partial x}+\frac{1-i\dot{G}}{2G}x\right)\phi(x,G(t))=0. \tag{19}$$

where $\phi(x,G(t))$ is just the ground-state wave function of $I(t)$, i.e., $|0\rangle$. We can easily solve Eq. (19) and get its normalized solution $\phi(x,G(t))$,

$$\phi(x,G(t))=(2\pi G)^{-1/4}\exp\left\{-\left(\frac{1-i\dot{G}}{2G}\right)\frac{x^2}{2}\right\}. \tag{20}$$

Then the solution of Eq. (13) is

$$\chi_{s=0}(x,t)=(2\pi G)^{-1/4}\exp\left\{-\frac{i}{4}\int^t\frac{1}{G(t')}dt'\right\}\times\exp\left\{-\left(\frac{1-i\dot{G}}{2G}\right)\frac{x^2}{2}\right\}. \tag{21}$$

At last, we must point out that the real wave function that relates to the movement of the quantum part of the system is $\chi(x,A)$, given by Eq. (15). It is easy to prove that the probability $|\chi(x,A(t)|^2$ equals to the $|\phi(x,G(t))|^2$, and so we will discuss only the wave function $\phi(x,G(t))$ instead of the $\chi(x,A(t))$.

IV. OBTAINED RESULTS AND DISCUSSION

In this section we will focus on the time evolution of the quantum part of the system and utilize the obtained wave functions and their Fourier analysis to show the difference and similarity between the so-called semiquantum chaotic state and the regular one. For simplicity, we will use the units of $\hbar=1$ and $m=1.0$. Given the fixed total energy $E=0.8$, we here use the initial conditions of Eqs. (17) and (18) as the following:

(i) Chaotic State: $A=0.0$; $G=0.5$; $\dot{A}=0.0$; $\dot{G}=0.774597$; $e=1.0$.

(ii) Regular State: $A=0.0$; $G=0.35$; $\dot{A}=0.0$; $\dot{G}=0.731925$; $e=1.0$.

These values are, respectively, picked up from the chaotic and regular region of the Poincaré section in the phase space, which was shown by Cooper *et al.* [6] in their paper. In order to safely identify whether the above states are, respectively, chaotic and regular ones, we have calculated their Lyapunov exponent (LE). It is found that the LE of the first state is positive, and that of the second one is zero, demonstrating quantitatively that the first state is indeed chaotic and the second is regular. After getting the values of $A(t)$ and $G(t)$ from the coupled equations of motion, we can use Eq. (20) to calculate the time evolution of the ground-state wave function in chaotic or regular states for the position x.

The Fourier spectra (FS) of the longtime evolution of the probability density $|\phi(x,G(t))|^2$ in chaotic and regular states are shown in Fig. 1. In it, we find that there exists a series of FS peaks at many frequencies in two states. At the frequency $\nu_1^R=0.4144$ Hz (the superscript R refers to the regular state) in the FS of the regular state, we have the maximal FS values. If ν_1^R is doubled, a peak of FS at 0.8289 Hz can be obtained, and if tripled there exists the peak at 1.2439 Hz, and so on. Besides the multiplier frequencies of the fundamental ν_1^R, we can discern some fractional frequencies of it, such as at $\nu_2^R=0.2075$ Hz and $\nu_3^R=0.0537$ Hz. Using ν_1^R

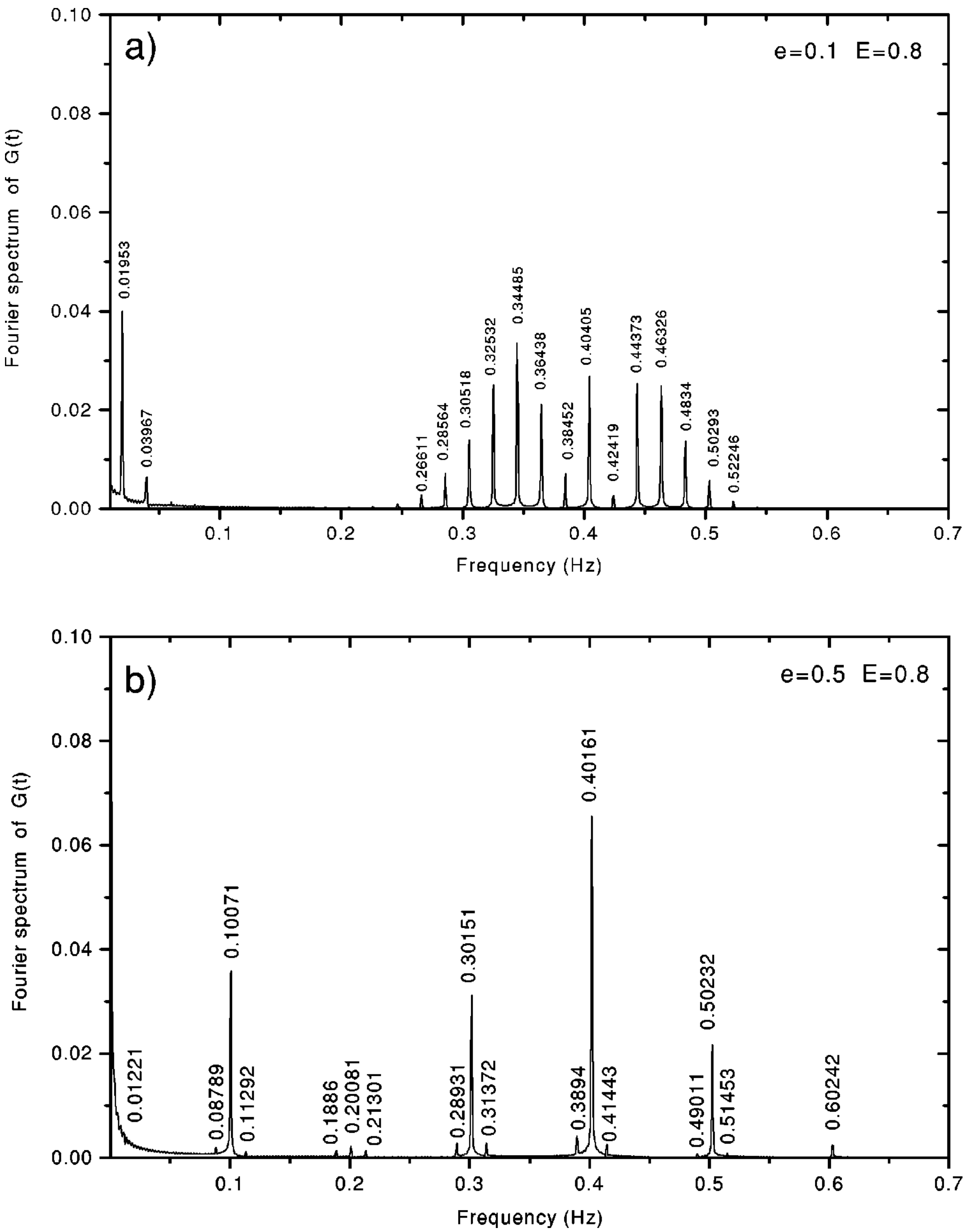

FIG. 2. Fourier spectra of $G(t)$ for different coupling coefficients. (a) $e=0.1$; (b) $e=0.5$.

and ν_2^R, we can obtain some combined frequencies in the FS like 0.6220 Hz and 1.036 Hz. As for other smaller FS components that appear on two sides of the bigger peaks mentioned above, we can easily get them through adding or subtracting the bigger frequencies by ν_3^R.

Now let us consider the FS of the chaotic state. At a glance of it, we can still distinguish the ''fundamental frequency'' from its FS, $\nu_1^C=0.4034$ Hz (the superscript C refers to the Chaotic state). Others such as $\nu_2^C=0.2020$ Hz and $\nu_3^C=0.0311$ Hz also play important roles in combining the frequencies in the FS, both of which are also the fractional frequencies of the ν_1^C in the 2nd and 13th order, respectively. For example, we can easily find that 0.6055 Hz is the sum of ν_1^C and ν_2^C. And at both sides of the main frequencies ν^C, there are also FS peaks at $\nu^C\pm\nu_3^C$, but now with unequal peak heights, which is different from the case of the regular state. For instance, the FS peak height at $\nu_4^C=\nu_1^C-\nu_3^C\approx 0.3705$ is much larger than that at frequency $\nu_5^C=\nu_1^C+\nu_3^C\approx 0.4364$, and even larger than that at ν_1^C, leading to the existence of two fundemental frequencies in the chaotic state, i.e., the $\nu_1^C=0.4034$ and $\nu_4^C=0.3705$. But if we analyze the FS carefully, we can find it has many differences from the regular one. First, there are much more new frequencies appearing in chaotic FS in comparison with the regular one. But their amplitudes are so small that they can hardly be selected out from the FS, which makes the FS curve be not smooth like that in the regular state. Second, the relative peak amplitudes at the frequencies change greatly. Obviously, the amplitude distribution in the chaotic FS seems to be more even than that in the regular one.

In order to answer the question of where the fundamental frequencies come from, we must investigate the coupled equations of motion for the coordinates $A(t)$ and $G(t)$, i.e., Eqs. (17) and (18). We solve the coupled equations when choosing $e=0.0$, $e=0.1$, $e=0.5$, and $e=1.0$. The initial conditions of the equations are chosen as the same as that of the regular state. It is easy to understand that when $e=0.0$, $A(t)$ undergoes a uniform motion and $G(t)$ oscillates with

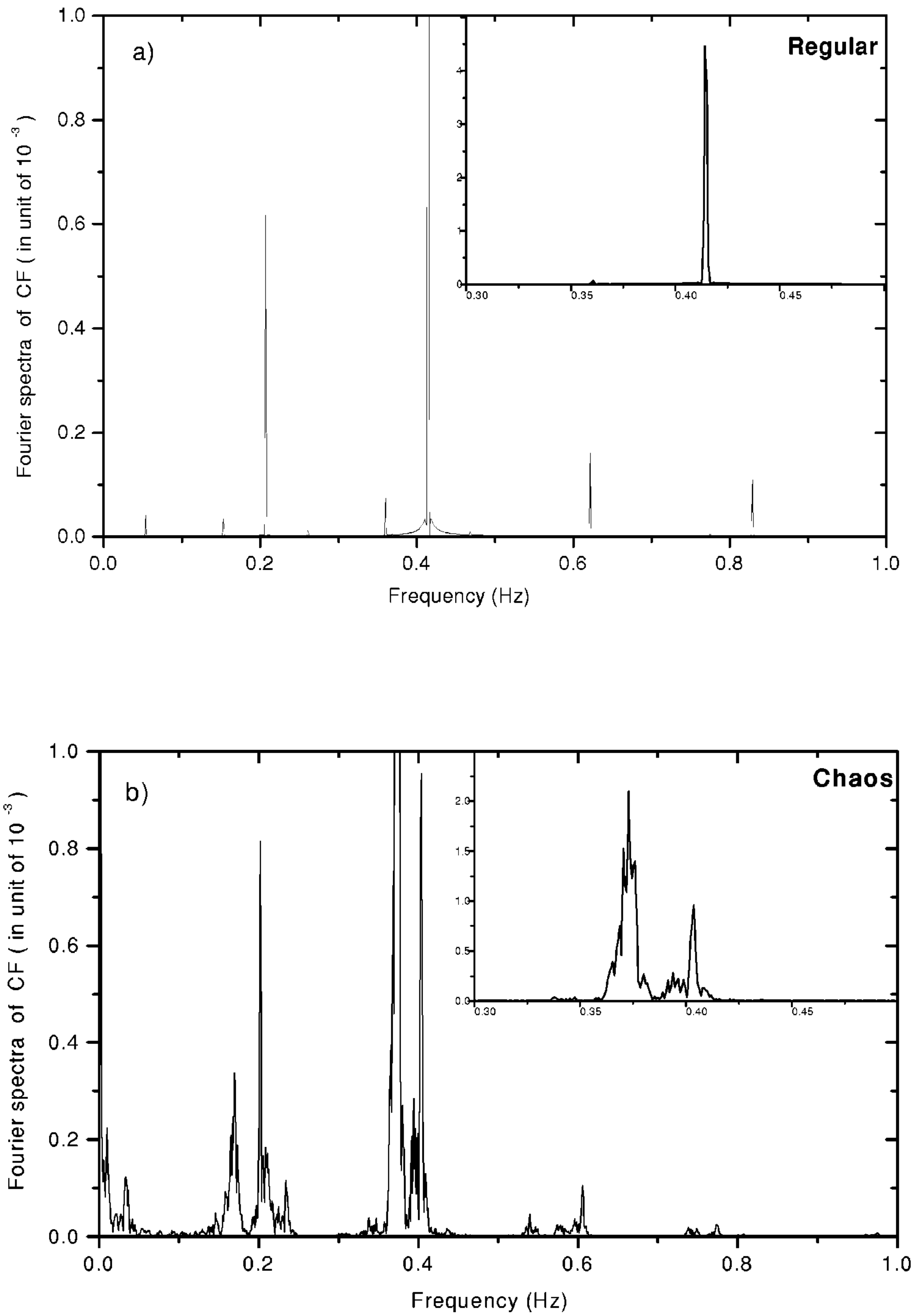

FIG. 3. Fourier spectra of the time-correlation functions of $|\phi(x,G(t))|$. (a) in regular state; (b) in chaotic state.

an unique frequency, 0.3183 Hz. When e changes from zero to 0.1, the coupled term in the equations plays an important role so that the motion of the system, especially $G(t)$, deviates from the simple pictures without the coupling. We show the FS of $G(t)$ at the energy $E=0.8$ in Fig. 2 for $e=0.1$ and $e=0.5$. The FS of $G(t)$ has many frequencies that lie around the center-frequency 0.404 05 Hz when $e=0.1$. While $e=0.5$, the FS of $G(t)$ has less frequencies but still centers at the almost unchanged frequency 0.4016 Hz. By comparing them with their counterpart of $e=1.0$, we realize that the FS of $G(t)$ has an intrinsic frequency at about 0.4 Hz for $e \neq 0$. With the increase of e, the coupling between $A(t)$ and $G(t)$ increases, causing the combination between the frequencies, which results in appearance of more frequencies in the FS. Now, we know that the fundamental frequencies in the FS of the wave functions originate from the coupling motion of the classical and quantum coordinates. And the pictures of the motion will greatly change when the system transforms from uncoupling to coupling.

Now, we study the correlation function of the time-dependent wave function. Given the same initial conditions, we get the time-correlation functions in the chaotic and regular state and their FS, respectively. We show the FS results in Fig. 3, in which the mean value of the FS has been filtered in order to remove the big dc-component in the FS. In the regular state [Fig. 3(a)], the FS of the correlation function is ''clear'' and it is easy to identify some fundamental frequencies, e.g., the $\nu_1^R, \nu_2^R, 2\nu_1^R, \nu_1^R+\nu_2^R$, etc. But when the system is in the chaotic state, the FS of the correlation function [Fig. 3(b)] becomes desultory, and it looks to have a lot of

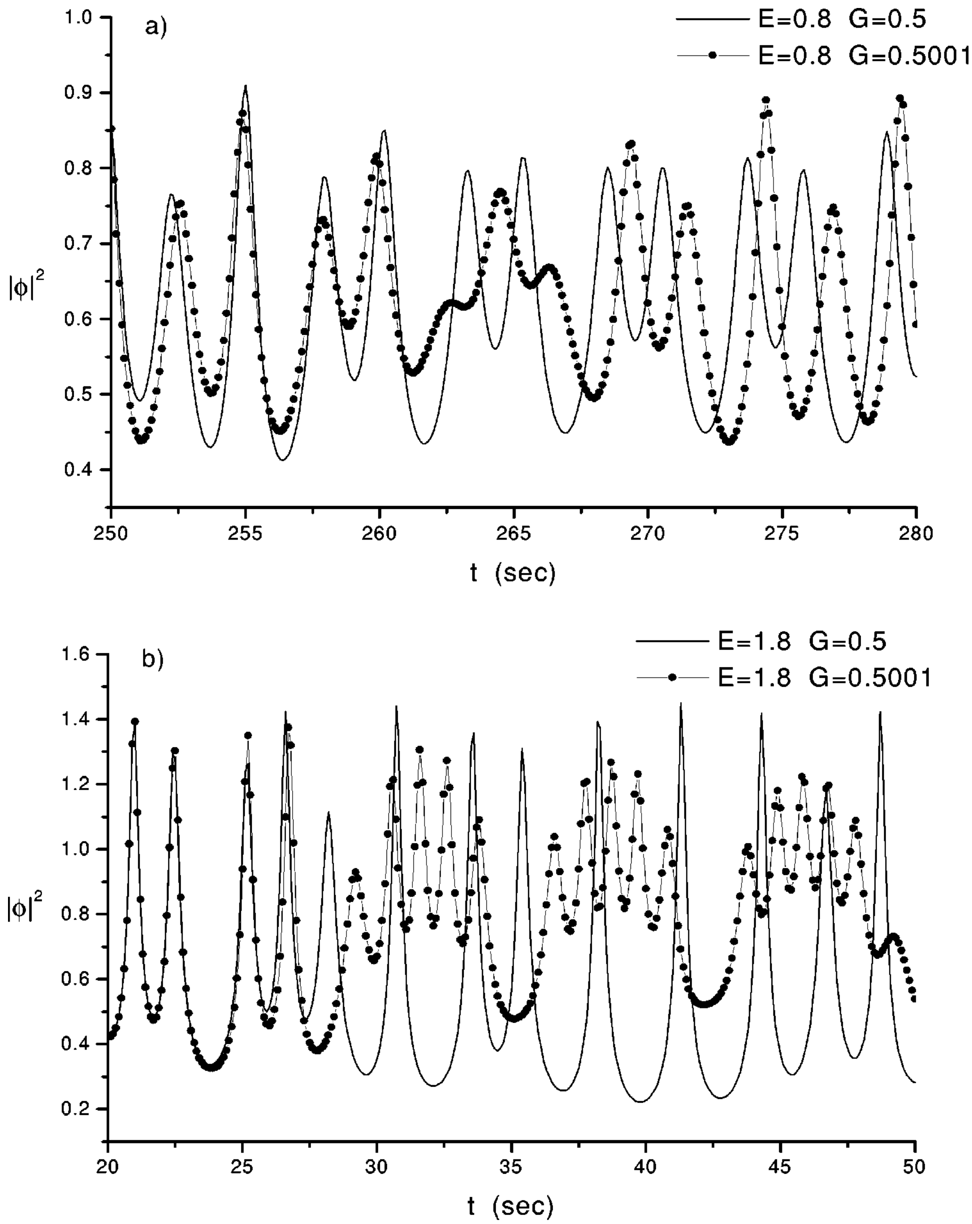

FIG. 4. Sensitivity of the probability density, $|\phi(x,G(t))|^2$, to the initial conditions for different energies. (a) $E=0.8$; (b) $E=1.8$.

''noise'' components. However, the fundamental frequencies still exist and other frequencies also appear because of a stronger nonlinear coupling effect existing in the chaotic state.

Finally, we examine what happens for the sensitivity of the eigenfunctions in the chaotic state to the initial conditions, which is usually considered as a basic characteristic of chaos. Figure 4 shows related numerical results of the probability density $|\phi|^2$ vs t for the case of $E=0.8$ [Fig. 4(a)] and 1.8 [Fig. 4(b)], which have the same initial conditions for all parameters except that the $G(0)$ has very little difference between two solutions, which are $G(0)=0.5$ and $G(0)=0.5001$. We can see from Fig. 4 that there indeed exists such sensitivity for the eigenfunction, which is infuluenced by the energy E. For lower energy $E=0.8$, the sensitivity is much weaker because only after a longer time (≈ 250 sec), can we identify the difference between two $|\phi|^2$ with almost the same $G(0)$. With E increasing, e.g., for $E=1.8$, the sensitivity becomes more clear and obvious because through only a very short time (≈ 28 sec), can two of the curves of $|\phi|^2$ show clear difference.

V. CONCLUSIONS

In this paper, we analytically calculate the wave function of a quantum oscillator coupled with a classical harmonic oscillator. Instead of using the definition of the classical chaos, such as the sensitivity of the system to the initial conditions, we have found some characteristics of the wave function in its time evolution and Fourier spectrum. We think these characteristics can be used to distinguish the regular and chaotic states when there emerges the so-called semiquantum chaos in the system. Our result can be thought of as another way that is accessible to understanding the semiquantum chaos.

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